Free Vibrations of Simply Supported Cylindrical Shells of Oval Cross Section

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A Flügge-type theory is employed in studying the dynamic characteristics of simply supported shells of oval cross section. The components of displacement are expressed in a double Fourier series of the axial and circumferential coordinates and substituted into the equations of motion. This leads to four different typical, algebraic, eigenvalue problems from which the frequencies and shapes of four types of harmonic modes are determined. These modes are either symmetric or antisymmetric with respect to the principal axes of the cross section of the shell. A comparison of the dynamic characteristics of oval cylindrical shells obtained on the basis of the Donnell, Love, Sanders, and Flügge theories is presented. It is shown that the deviations in the results of these theories increase with increasing noncircularity.

Nomenclature

$A_{m,n}, B_{m,n}, C_{m,n}, D_{m,n}, E_{m,n}, F_{m,n}$	= Fourier coefficients
D	$= (1/12)Eh^3/(-\nu^2)$
E	= Young's modulus
t	= time
u, v, w	= axial, circumferential, and radial components of displacement of a point on the median surface of a shell, respectively
U_m, V_m, W_m	= displacement functions for axial wave number m
δ_{ij}	= Kronecker delta
ϵ	= ovality parameter
ν	= Poisson's ratio

Introduction

ALTHOUGH cylindrical shells of noncircular cross section constitute components of submarines, aircraft, and various parts of industrial plants, their dynamic characteristics have not been investigated as extensively as those of cylindrical shells of circular cross section. This is because the displacement equations of motion for circular cylindrical shells have constant coefficients, while those for noncircular cylindrical shells have variable coefficients.

The free and forced vibrations of infinitely long and simply supported oval cylindrical shells were investigated by Klosner and Pohle[1] and Klosner[2,3] using a Love-type theory. A perturbation technique is used which gives accurate results for small values of the ovality parameter.

The free vibrations of simply supported, cylindrical shells of oval cross section were also investigated by Culberson and Boyd[4] using Love- and Donnell-type theories. The components of displacement are expressed in a double Fourier series, which satisfies the boundary conditions at the end cross sections of the shell.

The differential equations which describe the vibrations of noncircular cylindrical shells under the influence of initial stresses were derived by Elsbernd and Leissa[5] on the basis of the Sanders theory. Frequencies of simply supported cylindrical shells of oval cross section without initial stresses and with initial uniform axial stress were obtained using the Galerkin method. The components of displacement were expanded into a double Fourier series.

The free vibrations of cylindrical shells of oval cross section having various types of boundary conditions were investigated by Chen and Kempner[6,7] using Sanders- and Donnell-type theories. The components of displacement are expanded in an infinite series of the normal modes of the circular cylindrical shell, whose perimeter is equal to that of the oval shell.

The vibrations of simply supported cylindrical shells having any given doubly symmetric cross section have been investigated by Malkina[8] on the basis of a bending theory of shallow cylindrical shells, as well as a bending theory of nonshallow, cylindrical shells due to Novozhilov. Moreover, the vibrations and stability of anisotropic and sandwich cylindrical shells of any given cross section under lateral pressure have been investigated by Slepov.[9]

The accuracy of the dynamic characteristics of circular cylindrical shells obtained on the basis of the Donnell, Sanders, and Love theories (not the version used in Ref. 4) has been investigated extensively.[10,11] However, the accuracy of the dynamic characteristics of noncircular cylindrical shells obtained on the basis of these theories has not been investigated as yet. Therefore, in this investigation, the Flügge theory is employed in studying the dynamic characteristics of simply supported cylindrical shells of oval cross section by expanding the components of displacement into double Fourier series. Numerical results are presented and compared with those obtained on the basis of the Donnell, Love, and Sanders theories.

Displacement Equations of Motion

A thin-walled, closed, simply supported cylindrical shell of uniform thickness h made of an isotropic, linearly elastic material is considered. The shell is referred to a right-hand system of orthogonal curvilinear coordinates x^*, s^*, and z; x^* is measured along the axis of the shell, s^* along the curve formed by the intersection of the plane normal to the axis of the shell and its middle surface, and z inward along the direction perpendicular to the middle surface of the shell (see Fig. 1).

The x^* and s^* coordinates and the radius of curvature r^* of the cross section of the shell are nondimensionalized with respect to the radius r_0 of a circle whose perimeter is equal to that of the cross section of the noncircular shell. That is,

$$x = x^*/r_0 \qquad s = s^*/r_0 \qquad r = r^*/r_0 \tag{1}$$

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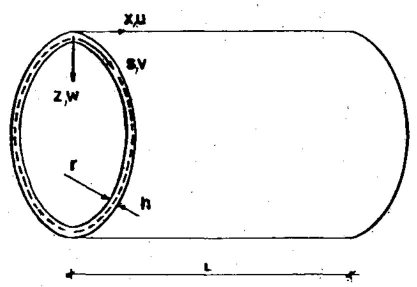

Fig. 1 Geometry of a cylindrical shell.

The displacement equations of equilibrium of a Flügge-type theory for cylindrical shells of any cross section subjected to a given distributed external force have been derived by Kempner.[12] From these, the following set of three homogeneous partial differential equations, describing the free vibrations of a cylindrical shell of any cross section, are obtained by adding the inertia terms

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u(x,s,t) \\ v(x,s,t) \\ w(x,s,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$L_{11} = \frac{\partial^2}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial s^2} + \frac{h^2}{12r_0^2}\left(\frac{1-\nu}{2}\right)\frac{1}{r^2}\frac{\partial^2}{\partial s^2} - \frac{h^2}{12r_0^2}\left(\frac{1-\nu}{2}\right)\frac{2r_{,s}}{r^3}\frac{\partial}{\partial s} - \frac{(1-\nu^2)}{E}\rho r_0^2\frac{\partial^2}{\partial t^2} \tag{3a}$$

$$L_{12} = L_{21} = \left(\frac{1+\nu}{2}\right)\frac{\partial^2}{\partial x \partial s} \tag{3b}$$

$$L_{13} = L_{31} = -\frac{\nu}{r}\frac{\partial}{\partial x} + \frac{h^2}{12r_0^2}\frac{1}{r}\frac{\partial^3}{\partial x^3} - \frac{h^2}{12r_0^2}\frac{1-\nu}{2}\frac{1}{r}\frac{\partial^3}{\partial x \partial s^2} + \frac{h^2}{12r_0^2}\frac{1-\nu}{2}\frac{r_{,s}}{r^2}\frac{\partial^2}{\partial x \partial s} \tag{3c}$$

$$L_{22} = \frac{\partial^2}{\partial s^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{h^2}{12r_0^2}3\left(\frac{1-\nu}{2}\right)\frac{1}{r^2}\frac{\partial^2}{\partial x^2} - \frac{h^2}{12r_0^2}\left(\frac{r_{,s}}{r^2}\right)^2 - \frac{(1-\nu^2)}{E}\rho r_0^2\frac{\partial^2}{\partial t^2} \tag{3d}$$

$$L_{23} = -\frac{1}{r}\frac{\partial}{\partial s} + \frac{r_{,s}}{r^2} + \frac{h^2}{12r_0^2}\left(\frac{3-\nu}{2}\right)\frac{1}{r}\frac{\partial^3}{\partial x^2 \partial s} + \frac{h^2}{12r_0^2}\frac{r_{,s}}{r^2}\frac{\partial^2}{\partial s^2} + \frac{h^2}{12r_0^2}\frac{r_{,s}}{r^4} \tag{3e}$$

$$L_{32} = -\frac{1}{r}\frac{\partial}{\partial s} + \frac{h^2}{12r_0^2}\left(\frac{3-\nu}{2}\right)\frac{1}{r}\frac{\partial^3}{\partial x^2 \partial s} - \frac{h^2}{12r_0^2}\left(\frac{3-\nu}{2}\right)\frac{r_{,s}}{r^2}\frac{\partial^2}{\partial x^2} - \frac{h^2}{12r_0^2}\frac{r_{,s}}{r^2}\frac{\partial^2}{\partial s^2} - \frac{2h^2}{12r_0^2}\left(\frac{r_{,s}}{r^2}\right)_{,s}\frac{\partial}{\partial s} - \frac{h^2}{12r_0^2}\left(\frac{r_{,s}}{r^2}\right)_{,ss} - \frac{h^2}{12r_0^2}\frac{r_{,s}}{r^4} \tag{3f}$$

$$L_{33} = \frac{h^2}{12r_0^2}\nabla^4 + \frac{h^2}{12r_0^2}\frac{1}{r^2}\frac{\partial^2}{\partial s^2} + \frac{1}{r^2} + \frac{h^2}{12r_0^2}\frac{1}{r^2}\frac{\partial^2}{\partial s^2} - \frac{4h^2}{12r_0^2}\frac{r_{,s}}{r^3}\frac{\partial}{\partial s} - \frac{2h^2}{12r_0^2}\left(\frac{r_{,s}}{r^3}\right)_{,s} + \frac{h^2}{12r_0^2}\frac{1}{r^4} + \frac{(1-\nu^2)}{E}\rho r_0^2\frac{\partial^2}{\partial t^2} \tag{3g}$$

and

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial s^2} + \frac{\partial^4}{\partial s^4} \tag{3h}$$

where $(\,)_{,s}$ denotes differentiation with respect to s and ρ is the mass density of the shell.

Harmonic Oscillations of Simply Supported Cylindrical Shells

The components of displacement of shells are periodic functions of the circumferential coordinate with period 2π. Consequently, they can be expanded in a complete double Fourier series in the axial and circumferential variables. Thus, for harmonic oscillation of cylindrical shells of noncircular cross section, the solution of Eqs. (2) can be assumed as

$$u(x,s,t) = e^{i\omega t}\sum_{m=0,1,2}^{\infty}[\cos\lambda_m x U_m(s) + \sin\lambda_m x \bar{U}_m(s)] \tag{4a}$$

$$v(x,s,t) = e^{i\omega t}\sum_{m=0,1,2}^{\infty}[\sin\lambda_m x V_m(s) - \cos\lambda_m x \bar{V}_m(s)] \tag{4b}$$

$$w(x,s,t) = e^{i\omega t}\sum_{m=0,1,2}^{\infty}[\sin\lambda_m x W_m(s) - \cos\lambda_m x \bar{W}_m(s)] \tag{4c}$$

where

$$U_m(s) = \sum_{n=0,1,2}^{\infty}(A_{m,n}\cos ns + B_{m,n}\sin ns)$$

$$\bar{U}_m(s) = \sum_{n=0,1,2}^{\infty}(\bar{A}_{m,n}\cos ns + \bar{B}_{m,n}\sin ns)$$

$$V_m(s) = \sum_{n=0,1,2}^{\infty}(C_{m,n}\cos ns + D_{m,n}\sin ns)$$

$$\bar{V}_m(s) = \sum_{n=0,1,2}^{\infty}(\bar{C}_{m,n}\cos ns + \bar{D}_{m,n}\sin ns)$$

$$W_m(s) = \sum_{n=0,1,2}^{\infty}(E_{m,n}\cos ns + F_{m,n}\sin ns)$$

$$\bar{W}_m(s) = \sum_{n=0,1,2}^{\infty}(\bar{E}_{m,n}\cos ns + \bar{F}_{m,n}\sin ns)$$

with

$$B_{m,0} \equiv \bar{B}_{m,0} \equiv D_{m,0} \equiv \bar{D}_{m,0} \equiv F_{m,0} \equiv \bar{F}_{m,0} \equiv 0 \tag{5}$$

In Eqs. (4) ω is the circular frequency of the vibration and

$$\lambda_m = m\pi r_0 / L \tag{6}$$

where L is the length of the shell.

The expressions for the components of displacement [Eqs. (4)] must satisfy the boundary conditions at the simply supported ends of the shell. That is

$$N_{xx} = v = w = M_{xx} = 0 \text{ at } x=0 \text{ and } x = L/r_0 \tag{7}$$

From the relation between the stress resultants and the components of displacement [Eq. (12)] it can be shown that the boundary conditions [Eq. (7)] are equivalent to

$$u_{,x} = v = w = w_{,xx} = 0 \text{ at } x=0 \text{ and } x = L/r_0 \tag{8}$$

Note that the terms involving $\sin\lambda_m x$ in Eq. (4a) and the terms involving $\cos\lambda_m x$ in Eqs. (4b) and (4c) do not satisfy the boundary conditions. If these terms are omitted, the space of solutions of the differential Eqs. (2) is not restricted, provided that the series of Eqs. (4) converge absolutely. Hence, for harmonic oscillations of simply supported shells the solution of Eqs. (2) takes the following form:

$$u(x,s,t) = e^{i\omega t} \sum_{m=0,1,2}^{\infty} \cos\lambda_m x U_m(s)$$

$$v(x,s,t) = e^{i\omega t} \sum_{m=1,2,3}^{\infty} \sin\lambda_m x V_m(s)$$

$$w(x,s,t) = e^{i\omega t} \sum_{m=1,2,3}^{\infty} \sin\lambda_m x W_m(s) \tag{9}$$

Substitution of Eqs. (9) into the equations of motion (2) for free vibrations of cylindrical shells yields

$$\sum_{m=0,1,2}^{\infty} P_m \cos\lambda_m x = 0 \tag{10a}$$

$$\sum_{m=1,2,3}^{\infty} Q_m \sin\lambda_m x = 0 \tag{10b}$$

$$\sum_{m=1,2,3}^{\infty} R_m \sin\lambda_m x = 0 \tag{10c}$$

where

$$P_m = -\left(\frac{1-\nu}{2}\right)\left[1+\frac{k}{r^2}\right]U_m''(s) + k\left(\frac{1-\nu}{2}\right)\frac{2r'}{r^3}U_m'(s)$$
$$+ [\lambda_m^2 - \Omega^2]U_m(s) - \left(\frac{1+\nu}{2}\right)\lambda_m V_m'(s) + k\left(\frac{1-\nu}{2}\right)\frac{\lambda_m}{r}$$
$$\times W_m''(s) - k\left(\frac{1-\nu}{2}\right)\lambda_m \frac{r'}{r^2} W_m'(s) + \frac{1}{r}[\nu\lambda_m + k\lambda_m^3]W_m(s)$$
$$\text{for } m=0,1,2,... \tag{11a}$$

$$Q_m = \left(\frac{1+\nu}{2}\right)\lambda_m U_m'(s) - V_m''(s) + \left[\left(\frac{1-\nu}{2}\right)\lambda_m^2 + k\left(\frac{r'}{r^2}\right)^2\right.$$
$$\left. + 3k\left(\frac{1-\nu}{2}\right)\frac{\lambda_m^2}{r^2} - \Omega^2\right]V_m(s) - k\frac{r'}{r^2}W_m''(s)$$
$$+\frac{1}{r}\left[1+k\left(\frac{3-\nu}{2}\right)\lambda_m^2\right]W_m'(s) - \left[\frac{r'}{r^2} - \frac{kr'}{r^4}\right]W_m(s)$$
$$\text{for } m=1,2,3... \tag{11b}$$

$$R_m = k\left(\frac{1-\nu}{2}\right)\frac{\lambda_m}{r}U_m''(s) - k\left(\frac{1-\nu}{2}\right)\lambda_m\frac{r'}{r^2}U_m'(s)$$
$$+\left[\frac{k\lambda_m^3}{r} + \frac{\nu\lambda_m}{r}\right]U_m(s) - \frac{kr'}{r^2}V_m''(s)$$
$$-\left[k\left(\frac{3-\nu}{2}\right)\frac{\lambda_m^2}{r} + \frac{1}{r} + 2k\left(\frac{r'}{r^2}\right)'\right]V_m'(s)$$
$$+\left[k\left(\frac{3-\nu}{2}\right)\lambda_m^2\frac{r'}{r^2} - k\left(\frac{r'}{r^2}\right)'' - \frac{kr'}{r^4}\right]V_m(s)$$
$$+kW_m''''(s) + \left[\frac{2k}{r^2} - 2k\lambda_m^2\right]W_m''(s) - 4\frac{kr'}{r^3}W_m'(s)$$
$$+\left[k\lambda_m^4 + 2k\left(\frac{r'}{r^3}\right)' + \frac{k}{r^4} + \frac{1}{r^2} - \Omega^2\right]W_m(s)$$
$$\text{for } m=1,2,3... \tag{11c}$$

In the above expressions, the prime denotes differentiation with respect to s; Ω^2 is the nondimensional frequency and k the nondimensional thickness parameter defined as

$$\Omega^2 = [(1-\nu^2)/E]\rho r_0^2\omega^2 \qquad k = h^2/12r_0^2 \tag{12}$$

Equations (10) are valid for any value of x, consequently the coefficients of the linearly independent sine or cosine terms must vanish. Thus, for any value of m greater than zero, the following set of three ordinary linear differential equations with variable coefficients is obtained for the functions $U_m(s)$, $V_m(s)$, and $W_m(s)$,

$$P_m = 0, \quad Q_m = 0, \quad R_m = 0 \text{ for } m=1,2,3... \tag{13}$$

For $m=0$, referring to Eqs. (9), it can be seen that the motion is purely longitudinal ($u\neq 0$, $v=w=0$). In this case, the function $U_0(s)$ is obtained from the following ordinary linear differential equation

$$P_0 = 0 \tag{14}$$

In order to proceed further in the solution of the system of ordinary differential Eqs. (13) or of Eq. (14), the radius of curvature $r(s)$ of the cross section of the cylindrical shell under consideration must be specified.

Harmonic Oscillations of Simply Supported Cylindrical Shells of Oval Cross Section

Romano and Kempner[13] defined an oval section as one whose curvature $1/r(s)$ is given by the first two terms of the Fourier series representation of the curvature of a general cylindrical shell used by Marguerre.[14] That is

$$1/r(s) = 1 + \epsilon\cos 2s \tag{15}$$

In order to prevent negative curvature, $|\epsilon|$ must be equal to or less than unity. In this range of ϵ $(0 \leq |\epsilon| \leq 1)$, the ratio of the major to minor axes of the oval varies from 1 to 2.06. For small values of $|\epsilon|$, $(|\epsilon| \leq ½)$, an oval approximates satisfactorily an ellipse having the same major to minor axes ratio.[13] Notice, that changing the sign of ϵ corresponds to a rigid body rotation of the shell about its axis of $\pi/2$ rad.

Substitution of Eq. (15) for the curvature into Eqs. (13) and (14) yields the following set of three homogeneous equations:

$$\sum_{n=0,1,2}^{\infty} G_{m,n}^{(1)} \cos ns + \sum_{n=1,2,3}^{\infty} H_{m,n}^{(1)} \sin ns = 0$$
$$(m=0,1,2...) \tag{16a}$$

$$\sum_{n=1,2,3}^{\infty} G_{m,n}^{(2)} \sin ns + \sum_{n=0,1,2}^{\infty} H_{m,n}^{(2)} \cos ns = 0$$
$$(m=1,2,3...) \tag{16b}$$

$$\sum_{n=0,1,2}^{\infty} G_{m,n}^{(3)} \cos ns + \sum_{n=1,2,3}^{\infty} H_{m,n}^{(3)} \sin ns = 0$$
$$(m=1,2,3...) \tag{16c}$$

where

$$G^{(1)}_{m,n} = (\alpha_1 - \Omega^2)A_{m,n} + \alpha_2 A_{m,n-2} + \alpha_3 A_{m,n+2} + \alpha_4 A_{m,n-4} + \alpha_5 A_{m,n+4} + \alpha_6 D_{m,n} + \alpha_7 E_{m,n} + \alpha_8 E_{m,n-2} + \alpha_9 E_{m,n+2} \tag{17a}$$

$$G^{(2)}_{m,n} = \alpha_{10} A_{m,n} + (\alpha_{11} - \Omega^2) D_{m,n} + \alpha_{12} D_{m,n-2} + \alpha_{13} D_{m,n+2} + \alpha_{14} D_{m,n-4} + \alpha_{15} D_{m,n+4} + \alpha_{16} E_{m,n} + \alpha_{17} E_{m,n-2} + \alpha_{18} E_{m,n+2} + \alpha_{19} E_{m,n-4} + \alpha_{20} E_{m,n+4} + \alpha_{21} E_{m,n-6} + \alpha_{22} E_{m,n-6} \tag{17b}$$

$$G^{(3)}_{m,n} = \alpha_{23} A_{m,n} + \alpha_{24} A_{m,n-2} + \alpha_{25} A_{m,n+2} + \alpha_{26} D_{m,n} + \alpha_{27} D_{m,n-2} + \alpha_{28} D_{m,n+2} + \alpha_{29} D_{m,n-4} + \alpha_{30} D_{m,n+4} + \alpha_{31} D_{m,n-6} + \alpha_{32} D_{m,n+6} + (\alpha_{33} - \Omega^2) E_{m,n} + \alpha_{34} E_{m,n-2} + \alpha_{35} E_{m,n+2} + \alpha_{36} E_{m,n-4} + \alpha_{37} E_{m,n+4} + \alpha_{38} E_{m,n-6} + \alpha_{39} E_{m,n+6} + \alpha_{40} E_{m,n-8} + \alpha_{41} E_{m,n+8} \tag{17c}$$

$$H^{(1)}_{m,n} = (b_1 - \Omega^2) B_{m,n} + b_2 B_{m,n-2} + b_3 B_{m,n+2} + b_4 B_{m,n-4} + b_5 B_{m,n+4} + b_6 C_{m,n} + b_7 F_{m,n} + b_8 F_{m,n-2} + b_9 F_{m,n+2} \tag{17d}$$

$$H^{(2)}_{m,n} - b_{10} B_{m,n} + (b_{11} - \Omega^2) C_{m,n} + b_{12} C_{m,n-2} + b_{13} C_{m,n+2} + b_{14} C_{m,n-4} + b_{15} C_{m,n+4} + b_{16} F_{m,n} + b_{17} F_{m,n-2} + b_{18} F_{m,n+2} + b_{19} F_{m,n-4} + b_{20} F_{m,n+4} + b_{21} F_{m,n-6} + b_{22} F_{m,n+6} \tag{17e}$$

$$H^{(3)}_{m,n} = b_{23} B_{m,n} + b_{24} B_{m,n-2} + b_{25} B_{m,n+2} + b_{26} C_{m,n} + b_{27} C_{m,n-2} + b_{28} C_{m,n+2} + b_{29} C_{m,n-4} + b_{30} C_{m,n+4} + b_{31} C_{m,n-6} + b_{32} C_{m,n+6} + (b_{33} - \Omega^2) F_{m,n} + b_{34} F_{m,n-2} + b_{35} F_{m,n+2} + b_{36} F_{m,n-4} + b_{37} F_{m,n+4} + b_{38} F_{m,n-6} + b_{39} F_{m,n+6} + b_{40} F_{m,n-8} + b_{41} F_{m,n+8} \tag{17f}$$

The parameters α_i and b_i are given in the Appendix. Note that there is no term (corresponding to $n=0$) involving the coefficients $H^{(1)}_{m0}$ and $H^{(3)}_{m0}$ in Eqs. (16a) and (16c). Moreover, there is no term involving the coefficients $G^{(2)}_{m0}$ in Eq. (16b).

In order that the three homogeneous Eqs. (16) vanish for any value of s, the coefficients of the linearly independent cosns and sinns terms must vanish ($G^{(j)}_{mn}=0$, $H^{(j)}_{mn}=0$, $j=1,2,3$). In these relations, $G^{(j)}_{mn}$ involves only the Fourier coefficients A_{mn}, D_{mn}, and E_{mn}, whereas $H^{(j)}_{mn}$ involves only the Fourier coefficients B_{mn}, C_{mn}, and F_{mn}. Thus, for any value of m, two independent solutions are obtained: one involving only the coefficients A_{mn}, D_{mn}, and E_{mn} and the other only the coefficients B_{mn}, C_{mn}, and F_{mn}. The two solutions are

$$U_m(s) = \sum_{n=0,1,2}^{\infty} A_{mn} \cos ns \qquad V_m(s) = \sum_{n=1,2,3}^{\infty} D_{mn} \sin ns \qquad W_m(s) = \sum_{n=0,1,2}^{\infty} E_{mn} \cos ns \tag{18}$$

and

$$U_m(s) = \sum_{n=1,2,3}^{\infty} B_{mn} \sin ns \qquad V_m(s) = \sum_{n=0,1,2}^{\infty} C_{mn} \cos ns \qquad W_m(s) = \sum_{n=1,2,3}^{\infty} F_{mn} \sin ns \tag{19}$$

Solution (18) is referred to as *symmetric* because the components of displacement u and w are symmetric with respect to the Y axis (see Fig. 2), while solution (19) is called *antisymmetric* because the components of displacement u and w are antisymmetric with respect to the Y axis.

Moreover, referring to Eqs. (17) it is apparent that $G^{(j)}_{mn}$ or $H^{(j)}_{mn}$ involve either only odd or only even n Fourier coef-

Table 1 Equations for four types of modes of vibrations of simply supported cylindrical shells of oval cross section

SS modes: Symmetric about X axis Symmetric about Y axis		SA modes: Symmetric about X axis Antisymmetric about Y axis	
$m=1,2,3...$ Even $n=2p-2$ $(p=2,3...)$ $G^{(j)}_{mn}=0$ $(j=1,2,3)$ $G^{(1)}_{m0}=G^{(3)}_{m0}=0$	(20)	$m=1,2,3...$ Odd $n=2p-1$ $(p=1,2,3...)$ $H^{(j)}_{mn}=0$ $(j=1,2,3)$	(26)
$m=0$ Even $n=2p-2$ $(p=2,3...)$ $G^{(1)}_{0n}=0$ $G^{(1)}_{00}=0$	(21)	$m=0$ Odd $n=2p-1$ $(p=1,2,3...)$ $H^{(1)}_{0n}=0$	(27)
AS modes: Antisymmetric about X axis Symmetric about Y axis		AA modes: Antisymmetric about X axis Antisymmetric about Y axis	
$m=1,2,3...$ Odd $n=2p-1$ $(p=1,2,3...)$ $G^{(j)}_{mn}=0$ $(j=1,2,3)$	(22)	$m=1,2,3...$ Even $n=2p-2$ $(p=2,3...)$ $H^{(j)}_{mn}=0$ $(j=1,2,3)$ $H^{(2)}_{mn}=0$	(24)
$m=0$ Odd $n=2p-1$ $(p=1,2,3...)$ $G^{(1)}_{0n}=0$	(23)	$m=0$ Even $n=2p-2$ $(p=2,3...)$ $H^{(1)}_{0n}=0$	(25)

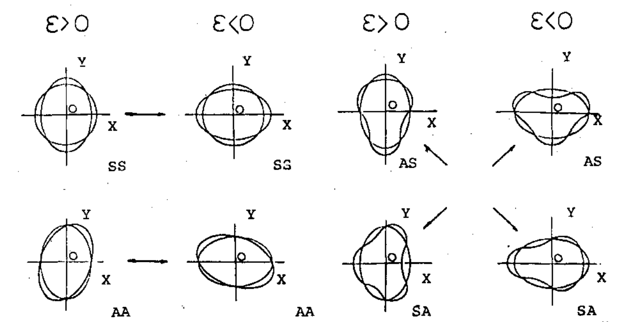

Fig. 2 Schematic representation of the different types of modes of oval cylindrical shells.

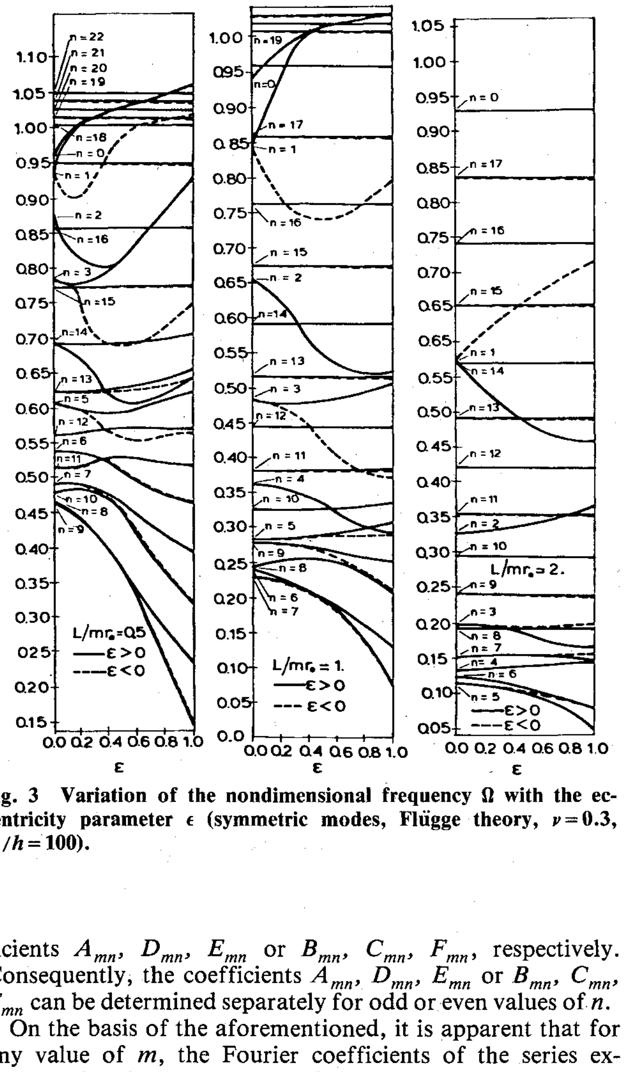

Fig. 3 Variation of the nondimensional frequency Ω with the eccentricity parameter ϵ (symmetric modes, Flügge theory, $\nu=0.3$, $r_0/h=100$).

ficients A_{mn}, D_{mn}, E_{mn} or B_{mn}, C_{mn}, F_{mn}, respectively. Consequently, the coefficients A_{mn}, D_{mn}, E_{mn} or B_{mn}, C_{mn}, F_{mn} can be determined separately for odd or even values of n.

On the basis of the aforementioned, it is apparent that for any value of m, the Fourier coefficients of the series expansion for the components of displacement [Eqs. (9)] are divided into four groups SS, AS, SA, and AA, each associated with one of the four types of uncoupled natural modes of vibration listed in Table 1. The Fourier coefficients belonging to the same group are coupled in an infinite set of homogeneous, linear algebraic equations, which depend on the nondimensional frequency Ω^2. In order to establish the eigenfrequencies and the corresponding mode shapes, the Fourier series expansions [Eqs. (9)] are truncated, retaining only the terms corresponding to $p=1,2,...P$. The value of P is chosen so that the frequencies and mode shapes are established to within the desired accuracy. Referring to Eqs. (20), (22), (24), and (26), it is apparent that, for any value of m greater than zero, the Fourier coefficients for the symmetric SS and AS and the antisymmetric AA and SA modes are established from a set of $3P-1$, $3P$, $3P-2$, and $3P$ homogeneous linear algebraic equations, respectively.

Moreover, referring to Eqs. (21), (23), (25), and (27) it is apparent that for $m=0$ the Fourier coefficients for each of the SS, AS, AA, SA modes are established from a set of P linear algebraic equations of the form

$$G_{0n}^{(1)} = (\alpha_1-\Omega^2)A_{0n}+\alpha_2 A_{0n-2}+\alpha_3 A_{0n+2} +\alpha_4 A_{0n-4}+\alpha_5 A_{0n+4}=0 \qquad (28)$$

and

$$H_{0n}^{(1)} = (b_1-\Omega^2)B_{0n}+b_2 B_{0n-2}+b_3 B_{0n+2} +b_4 B_{0n-4}+b_5 B_{0n+4}=0 \qquad (29)$$

These modes are purely longitudinal and independent of the axial coordinate. Referring to the expressions for α_i and b_i given in the Appendix, it is observed that for $n=0$, Eq. (28) gives $\Omega^2=0$. That is, the motion is a rigid-body translation of the shell along its axis. This motion is permitted by the given boundary conditions [Eq. (8)].

In order to write Eqs. (20-29) in a compact form we introduce the following vectors of unknown Fourier coefficients:

$$X^{(1)T}=[A_{m0}E_{m0}A_{m2}D_{m2}E_{m2}A_{m4}\cdots A_{m2P-2}D_{m2P-2}E_{m2P-2}]$$

$$X^{(2)T}=[A_{m1}D_{m1}E_{m1}A_{m3}D_{m3}E_{m3}\cdots A_{m2P-1}D_{m2P-1}E_{m2P-1}]$$

$$X^{(3)T}=[C_{m0}B_{m2}C_{m2}F_{m2}B_{m4}C_{m4}F_{m4}\cdots B_{m2P-2}C_{m2P-2}F_{m2P-2}]$$

$$X^{(4)T}=[B_{m1}C_{m1}F_{m1}B_{m3}C_{m3}F_{m3}\cdots B_{m2P-1}C_{m2P-1}F_{m2P-1}]$$

$$X^{(5)T}=[A_{02}A_{04}A_{06}\cdots A_{02P}]$$

$$X^{(6)T}=[A_{01}A_{03}A_{05}\cdots A_{02P-1}]$$

$$X^{(7)T}=[B_{02}B_{04}B_{06}\cdots B_{02P}]$$

$$X^{(8)T}=[B_{01}B_{03}B_{05}\cdots B_{02P-1}] \qquad (30)$$

Thus, for any value of m, the coefficients of the Fourier series expansions for the components of displacement [Eqs. (9)] are established from the following eight sets of homogeneous linear algebraic equations:

$$[M^{(q)}-\Omega^2 I]X^{(q)}=0 \quad (q=1\text{-}8) \qquad (31)$$

where for any value of m greater than zero, the SS, AS, AA, or SA modes are obtained for $q=1,2,3,4$, respectively; and I is the unit matrix and $M^{(q)}$ $(q=1,2,3,4)$ are real square matrices of order $3P-1$, $3P$, $3P-2$, and $3P$, respectively. Morever, for $m=0$ the SS, AS, AA, or SA axial modes are obtained for $q=5,6,7,8$, respectively. Referring to the expressions for the α_i and b_i given in the Appendix, it is observed that for $m=0$, they are related by the relations

$$\alpha_3(n)=\alpha_2(n+2), \qquad \alpha_5(n)=\alpha_4(n+4)$$

and

$$b_3(n)=b_2(n+2), \qquad b_5(n)=b_4(n+4) \qquad (32)$$

Thus, the matrices $M^{(q)}$ $(q=5,6,7,8)$ are real symmetric matrices of order P.

On the basis of the aforementioned, it is apparent that the problem of harmonic vibration of cylindrical shells of oval cross section reduces to a number of typical algebraic eigenvalue problems [Eq. (31) for $m=0,1,2,3...$]. Hence, in order to obtain a nontrivial solution $X^{(q)}$ $(q=1\text{-}8)$, the determinant of the coefficients of $X^{(q)}$ must vanish, resulting

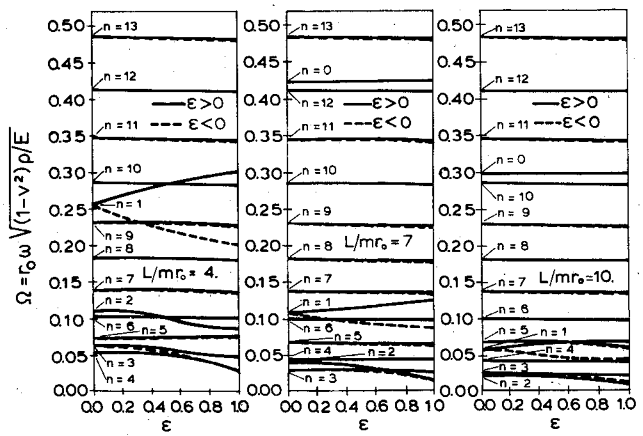

Fig. 4 Variation of the nondimensional frequency Ω with the eccentricity parameter ϵ (symmetric modes, Flügge theory, $\nu = 0.3$, $r_0/h = 100$).

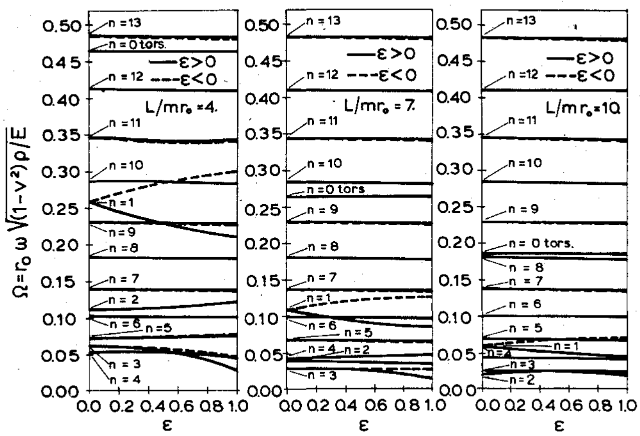

Fig. 5 Variation of the nondimensional frequency Ω with the eccentricity parameter ϵ (antisymmetric modes, Flügge theory, $\nu = 0.3$, $r_0/h = 100$).

Table 2 Nondimensional frequencies Ω, $\nu = 0.3$, $r_0/h = 100$

Mode	L/mr_0	n	ε=0 S≡A	ε=0.2 S	ε=0.2 A	ε=0.4 S	ε=0.4 A	ε=0.6 S	ε=0.6 A	ε=0.8 S	ε=0.8 A	ε=1.0 S	ε=1.0 A
1st	0.5	9	.46247	.44087	.44087	.39829	.39833	.33465	.33467	.24119	.24119	.15704	.15703
	1.0	7	.22852	.22293	.22293	.20798	.20768	.18218	.18206	.13757	.13770	.07618	.07618
	2.0	5	.11227	.11062	.11060	.10577	.10533	.09790	.09485	.07609	.07531	.04585	.04605
	4.0	4	.05547	.05497	.05500	.05316	.05364	.04901	.05039	.04101	.04017	.02928	.02757
	7.0	3	.02987	.02968	.02961	.02923	.02863	.02862	.02660	.02600	.02327	.01940	.01888
	10.0	2	.02195	.02152	.02196	.02026	.02197	.01824	.02199	.01563	.02010	.01275	.01714
2nd	0.5	8	.46555	.44113	.44114	.39912	.39908	.34637	.34634	.28842	.28841	.23527	.23526
	1.0	6	.23895	.22749	.22750	.20904	.20940	.18608	.18636	.15943	.15937	.13143	.13129
	2.0	6	.12234	.11905	.11907	.11108	.11169	.09819	.10208	.08869	.09118	.07834	.07989
	4.0	3	.06237	.06136	.06114	.05891	.05727	.05576	.05158	.05229	.04905	.04875	.04626
	7.0	2	.04145	.04010	.04093	.03682	.03990	.03201	.03873	.02792	.03753	.02718	.03632
	10.0	3	.02421	.02411	.02406	.02387	.02349	.02354	.02224	.02316	.02201	.02276	.02205
3rd	0.5	10	.47968	.48659	.48660	.47191	.47175	.42019	.41991	.36435	.36431	.32511	.32511
	1.0	8	.24367	.25068	.25068	.25463	.25496	.25044	.25463	.23238	.23183	.21238	.21120
	2.0	4	.13245	.13394	.13401	.13564	.13651	.13544	.13882	.13334	.14100	.13143	.14285
	4.0	5	.07314	.07354	.07353	.07448	.07436	.07565	.07514	.07689	.07579	.07819	.07648
	7.0	4	.04405	.04445	.04455	.04497	.04555	.04539	.04667	.04579	.04781	.04627	.04893
	10.0	4	.04263	.04257	.04257	.04236	.04237	.04201	.04204	.04151	.04159	.04088	.04103
4th	0.5	7	.49080	.49150	.49147	.47511	.47782	.44833	.44863	.42090	.42111	.39775	.39788
	1.0	9	.27730	.27712	.27710	.27246	.27136	.26503	.25654	.25720	.25112	.25061	.24677
	2.0	7	.15098	.15125	.15125	.15191	.15187	.15261	.15205	.15318	.14895	.15369	.14329
	4.0	6	.10258	.10253	.10253	.10240	.10240	.09761	.10218	.09115	.10187	.08835	.10149
	7.0	5	.06888	.06880	.06880	.06858	.06857	.06820	.06817	.06767	.06754	.06699	.06669
	10.0	1	.05910	.06176	.05620	.06410	.05327	.06597	.05067	.06641	.04884	.06569	.04809
5th	0.5	11	.51422	.51965	.51965	.52535	.52567	.49650	.49989	.47685	.47822	.46293	.46364
	1.0	5	.28124	.28203	.28198	.28671	.28595	.29249	.28893	.29823	.28930	.29146	.28879
	2.0	8	.19064	.19066	.19066	.19070	.18739	.19076	.17532	.19085	.16479	.19094	.16311
	4.0	2	.11303	.11095	.11358	.10522	.11514	.10219	.11757	.10189	.12066	.10152	.12423
	7.0	6	.10034	.10026	.10026	.10000	.09886	.09957	.09404	.09898	.09063	.09822	.08936
	10.0	5	.06829	.06821	.06821	.06796	.06795	.06768	.06754	.06836	.06699	.06975	.06634

Table 3 Percent differences in the frequencies of the first mode obtained on the basis of the Donnell, Love, and Sanders theories as compared to those obtained on the basis of the Flügge theory

r_o/h	$L/n\pi r_o$	n	ε=0.0 Donnell S≡A	ε=0.0 Love S≡A	ε=0.0 Sanders S≡A	ε=0.5 Donnell S	ε=0.5 Donnell A	ε=0.5 Love S	ε=0.5 Love A	ε=0.5 Sanders S	ε=0.5 Sanders A	ε=1.0 Donnell S	ε=1.0 Donnell A	ε=1.0 Love S	ε=1.0 Love A	ε=1.0 Sanders S	ε=1.0 Sanders A
20.	0.5	3	.59	.29	0.22	.22	.22	.13	.13	.12	.12	.01	.02	-0.01	-0.0005	-0.01	0.005
	1.0	4	2.12	1.43	0.24	1.03	.68	.66	.38	.12	.10	.03	.13	-0.09	-0.01	-0.1	0.004
	2.0	3	3.73	3.13	0.24	3.26	3.08	2.81	2.65	.21	.19	-0.57	1.74	-0.66	1.59	-0.33	0.09
	4.0	2	4.67	4.59	0.19	4.18	4.05	4.58	4.56	-0.09	.17	2.74	2.97	4.38	3.57	-1,40	-0.01
	7.0	2	17.36	18.71	0.31	16.67	18.75	19.74	22.48	-0.90	.27	12.11	16.73	17.67	21.28	-3,98	-0.27
	10.0	2	26.72	29.41	0.25	24.22	19.13	29.19	24.87	-1,63	.20	15.11	-1.26	22.20	14.18	-5,13	-3,35
	15.0	1	6.79	12.40	0.07	3.81	9.95	17.62	29.85	-6.87	-6.82	2.15	12.22	41.17	67.18	-32,46	-35,85
	20.0	1	18.89	34.06	0.15	11.83	29.54	47.45	77.32	-22,65	-22,68	7.84	45.81	99.77	162.11	-	-
	50.0	1	298.89	453.61	0.99	222.84	415.37	570.12	805.19	-*	-	173.52	593.58	958.35	1407.10	-	-
	100.	1	1438.80	2072.43	3.96	1125.39	1914.08	2542.98	3487.70	-	-	918.55	2636.03	3710.27	5896.92	-	-
100.	0.5	9	.49	.35	0.05	.17	.17	.11	.11	0.03	0.03	-0.001	.005	-0.01	-0.003	-0.01	-0.0003
	1.0	7	.97	.82	0.06	.42	.49	.34	.40	0.03	0.03	0.009	.03	-0.01	.004	-0.03	0.0003
	2.0	5	1.77	1.65	0.06	1.30	1.26	1.21	1.17	0.05	0.04	0.052	.25	0.02	.22	-0.07	0.01
	4.0	4	4.10	4.07	0.06	3.48	4.19	3.52	4.22	0.04	0.05	.749	.70	.83	.86	-0.23	0.01
	7.0	3	7.13	7.37	0.06	7.03	6.83	7.57	7.33	0.02	0.02	-1.538	4.90	-0.77	5.67	-0.79	-0.03
	10.0	2	4.88	5.42	0.04	4.40	14.60	5.57	15.45	-0.33	0.04	2.851	4.59	5.77	5.56	-2.08	-0.07
	15.0	2	15.03	16.83	0.07	14.58	16.01	18.03	19.95	-1.06	0.04	10.956	18.86	17.02	26.58	-4.24	-0.41
	20.0	2	23.88	26.77	0.06	22.05	26.57	27.12	32.85	-1.68	0.01	14.315	15.94	21.65	26.32	-5,23	-0.89
	50.0	1	26.36	47.87	0.04	16.91	41.15	65.88	105.14	-36,66	-37.10	11.490	64.49	133.48	213.86	-	-
	100.	1	223.97	345.50	0.16	163.81	314.34	437.75	624.29	-	-	125.742	454.57	746.24	1103.57	-	-

*Note: a dash indicates imaginary frequencies.

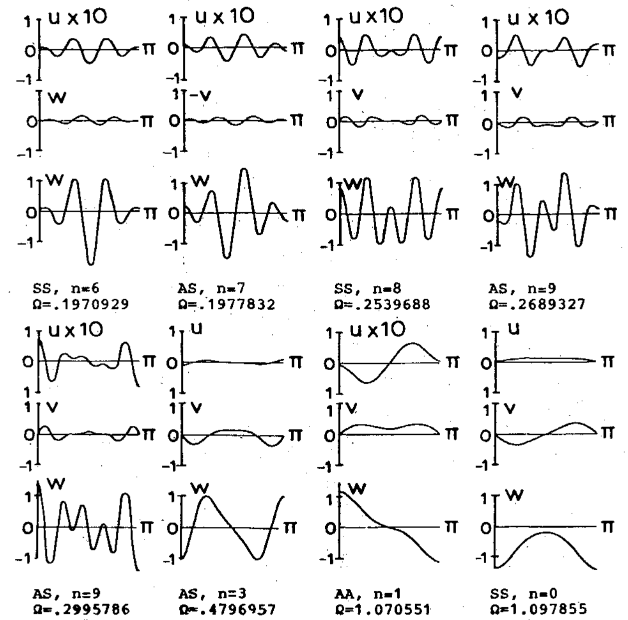

Fig. 6 Symmetric modes, Flügge theory ($\epsilon = 0.5$, $L/mr_0 = 1.0$, $\nu = 0.3$, $r_0/h = 100$).

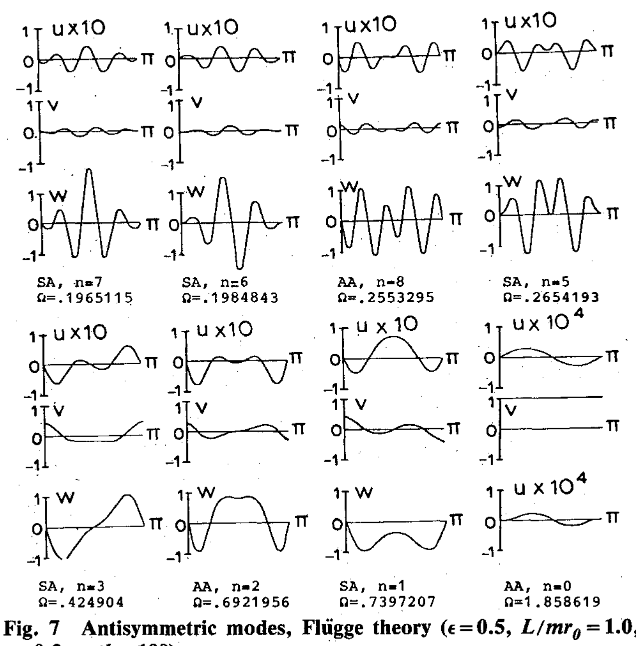

Fig. 7 Antisymmetric modes, Flügge theory ($\epsilon = 0.5$, $L/mr_0 = 1.0$, $\nu = 0.3$, $r_0/h = 100$).

in the following frequency equations:

$$\det[M^{(q)} - \Omega^2 I] = 0 \qquad q = 1,2,\ldots 8 \tag{33}$$

For every eigenfrequency obtained from Eq. (33), the corresponding eigenvector $X^{(q)}$ is determined from Eq. (31) and substituted into Eq. (18) for symmetric motions and Eq. (19) for antisymmetric motions to yield the associated mode shapes.

Numerical Results

The eigenvalues and the eigenvectors of the algebraic eigenvalue problems determined above are established using a computer program written for this purpose. This program employs a QR method for computing the eigenvalues and an inverse power method for determining the eigenvectors.[15]

For given values of r_0/h and ν the frequencies $\Omega[(L/mr_0),\epsilon]$ represent four infinite sets of continuous

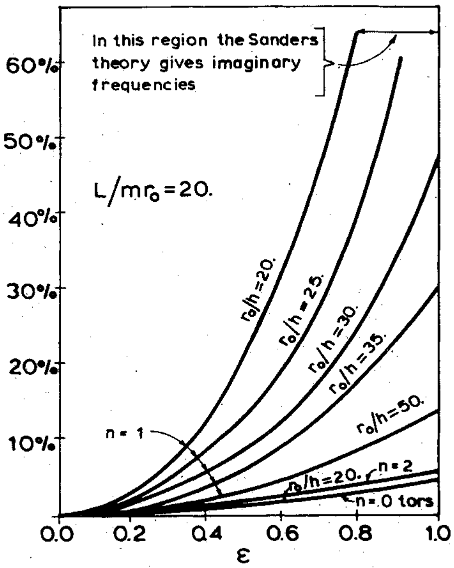

Fig. 8 Percent difference of frequencies obtained on the basis of the Sanders theory as compared to those obtained on the basis of the Flügge theory.

piecewise smooth frequency surfaces in the L/mr_0 and ϵ space. Each set of surfaces contains the frequencies of either the SS, AS, AA, or SA modes. Note that two surfaces of a set could touch or intersect along certain lines. Moreover, frequencies associated with the same frequency surface do not necessarily correspond to modes having the same number of circumferential nodes. Thus, it is possible to have the so-called dual-frequency phenomenon.[5,17]

The curves of intersection of the frequency surfaces for the SS, AS, AA, and SA modes with the planes $L/mr_0 = 0.5$, 1.0, 2.0, 4.0, 7.0, and 10.0 are shown in Figs. 3-5 for shells with $\nu = 0.3$ and $r_0/h = 100$. We will characterize the frequency lines in the plane $L/mr_0 = \text{const}$ by the value of the number of circumferential waves n associated with the modes of vibration of shells of circular cross section ($\epsilon = 0$).

In Figs. 3 and 4 the frequency lines with even values of n represent SS modes, while the frequency lines with odd values of n represent AS modes. Moreover, in Fig. 5 the frequency lines with even values of n represent AA modes, while the frequency lines with odd values of n represent SA modes.

In Figs. 3-5, the continuous lines represent the frequency curves for a shell with $\epsilon > 0$, while the broken lines represent the frequency curves for a shell with $\epsilon < 0$. Moreover, note that the frequency lines of the SS and AA modes coincide for $\epsilon > 0$ and $\epsilon < 0$ (Fig. 2).

In Table 2, the nondimensionalized frequencies Ω of the first five symmetric (SS and AS) and antisymmetric (AA and SA) modes are tabulated for shells with $r_0/h = 100$ and $\nu = 0.3$ for various values of L/mr_0 and ϵ.

In Table 3, the percent differences in the lower frequencies of shells with $r_0/h = 20$ and 100, obtained on the basis of the Donnell Love, and Sanders theories, compared to those obtained on the basis of the Flügge theory is tabulated for various values of L/mr_0 and for $\epsilon = 0.0$, 0.5, and 1.0.

Conclusions

From the analysis and the numerical results presented in this investigation, the following conclusions can be deduced:

1) For any value of m, the natural modes of vibration of simply supported cylindrical shells of oval cross section may be classified as SS, AS, SA, and AA modes, where the first S or A letter indicates symmetry or antisymmetry, respectively, with respect to the X axis, while the second S or A letter indicates symmetry or antisymmetry, respectively, with respect to the Y axis of the oval cross section of the shell. The SS and

AA modes are associated with the even harmonics, while the SA and AS modes are associated with the odd harmonics of the Fourier series expansions for $U_m(s)$, $V_m(s)$, $W_m(s)$ [Eq. (5)].

2) For $m=0$, the modes are purely longitudinal ($u\neq 0$, $w=v=0$) and independent of the axial coordinate. Moreover, in this case, for $n=0$ the shell moves as a rigid body ($\Omega^2=0$, $U_0=C$). That is, its boundary conditions allow movement as a rigid body in the axial direction.

3) Cylindrical shells of circular cross section can vibrate in purely torsional modes [$u=w=0$, $v\neq 0$]. However, this is not valid for cylindrical shells of oval cross section. This becomes apparent by noting that in the recurrence relations $G_{mn}^{(2)}=0$ or $H_{mn}^{(2)}=0$ the coefficients of the Fourier harmonics are coupled. Nevertheless, as evident from Fig. 2, when cylindrical shells of noncircular cross section vibrate in modes corresponding to the torsional modes of cylindrical shells of circular cross section, their u and w components of displacement are very small compared to their v component. Thus, these modes are almost purely torsional.

4) For cylindrical shells of circular cross section the frequencies of the modes with $n=0$ (torsional and nontorsional) and with $n=1$ do not vary with the r_0/h ratio. Moreover, for $n>1$ the effect of the r_0/h ratio on the frequencies is larger for bigger values of n and L/mr_0. For cylindrical shells of oval cross section, the variation with r_0/h of the frequencies corresponding to $n=0$ and 1 is negligible for $L/mr_0<20$.

5) As can be seen from Fig. 2, the SS and AA modes of shells with $\epsilon<0$ coincide with the corresponding modes of shells with $\epsilon>0$. Moreover, the frequency lines for the AS modes of shells with $\epsilon>0$ coincide with those for the SA modes of shells with $\epsilon<0$. Furthermore, notice that parts of some frequency curves for the AS and SA modes coincide with parts of neighboring frequency curves for the SS or AA modes, respectively.

6) The absolute value of the percent difference in the frequencies of the corresponding symmetric and antisymmetric modes is very small for values of $L/mr_0<1.0$.

7) The absolute value of the percent difference in the frequencies of the modes of vibration of cylindrical shells of oval cross section and the corresponding frequencies of cylindrical shells of circular cross section can reach 85% for shells with $r_0/h=2000$.[16]

8) The frequencies and mode shapes of the purely longitudinal modes of simply supported cylindrical shells of oval cross section obtained on the basis of the Donnell theory are almost identical to those obtained on the basis of the Flügge theory.

9) As anticipated, the Donnell theory for long and very long shells vibrating in modes with small circumferential wave numbers is highly inaccurate.

10) The version of Love theory used in Ref. 4 is more inaccurate than the Donnell-type theory for long and very long shells (see Table 3).

11) The error in the frequencies obtained on the basis of the Sanders theory increases as the ovality of the shell increases. For thin shells ($r_0/h>100$) vibrating in modes with a large number of circumferential nodes ($n\geq 3$), the frequencies obtained on the basis of the Sanders theory are very accurate. For thick, long, highly oval shells vibrating in the flexural mode ($n=1$), the frequencies obtained on the basis of the Sanders theory are either imaginary or highly inaccurate (see Fig. 8). However, in this range of shell parameters, the frequencies of the flexural mode of a shell may be approximated by those of an equivalent beam.

12) The numerical results presented in Refs. 5-7 are in excellent agreement with those obtained in this investigation on the basis of the Sanders theory. Moreover, for the range of shell parameters used in these references the error of the results obtained on the basis of the Sanders theory does not exceed 4%.

Appendix

$$\alpha_1=\left(\frac{1-\nu}{2}\right)n^2+kn^2\left(\frac{1-\nu}{2}\right)\left(1+\frac{\epsilon^2}{2}\right)+\lambda_m^2$$

$$-k\left(\frac{1-\nu}{2}\right)\epsilon\delta_{1n}-k\left(\frac{1-\nu}{2}\right)\epsilon^2\delta_{2n}$$

$$\alpha_2=k\left(\frac{1-\nu}{2}\right)\left[\epsilon(n-2)n-\frac{3}{4}\epsilon^2\delta_{3n}\right]$$

$$\alpha_3=k\left(\frac{1-\nu}{2}\right)\left[\epsilon(n+2)n-\frac{3}{4}\epsilon^2\delta_{1n}\right]$$

$$\alpha_4=k\left(\frac{1-\nu}{2}\right)\frac{\epsilon^4}{4}(n-4)n$$

$$\alpha_5=k\left(\frac{1-\nu}{2}\right)\frac{\epsilon^2}{4}(n+4)n$$

$$\alpha_6=-\left(\frac{1+\nu}{2}\right)\lambda_m n$$

$$\alpha_7=\nu\lambda_m+k\lambda_m^3-k\left(\frac{1-\nu}{2}\right)\lambda_m n^2$$

$$+\left[\frac{\nu\epsilon}{2}\lambda_m+k\lambda_m^3\frac{\epsilon}{2}+k\left(\frac{1-\nu}{2}\right)\lambda_m\frac{\epsilon}{2}\right]\delta_{1n}$$

$$\alpha_8=\frac{\nu\lambda_m\epsilon}{2}+k\lambda_m^3\frac{\epsilon}{2}-k\left(\frac{1-\nu}{2}\right)\lambda_m(n-2)n\frac{\epsilon}{2}$$

$$+\left(\frac{\nu\epsilon\lambda_m}{2}+k\lambda_m^3\frac{\epsilon}{2}\right)\delta_{2n}$$

$$\alpha_9=\frac{\nu\lambda_m\epsilon}{2}+k\lambda_m^3\frac{\epsilon}{2}-k\left(\frac{1-\nu}{2}\right)\lambda_m(n+2)n\frac{\epsilon}{2}$$

$$\alpha_{10}=-\left(\frac{1+\nu}{2}\right)\lambda_m n$$

$$\alpha_{11}=n^2+\left(\frac{1-\nu}{2}\right)\lambda_m^2+3k\left(\frac{1-\nu}{2}\right)\lambda_m^2\left(1+\frac{\epsilon^2}{2}\right)$$

$$+2k\epsilon^2-3k\left(\frac{1-\nu}{2}\right)\lambda_m^2\epsilon\left(\delta_{1n}+\frac{\epsilon}{4}\delta_{2n}\right)+k\epsilon^2\delta_{2n}$$

$$\alpha_{12}=3k\left(\frac{1-\nu}{2}\right)\lambda_m^2\epsilon\left(1-\delta_{2n}-\frac{\epsilon}{4}\delta_{3n}\right)+k\epsilon^2\delta_{3n}$$

$$\alpha_{13}=3k\left(\frac{1-\nu}{2}\right)\lambda_m^2\epsilon\left(1-\frac{\epsilon}{4}\delta_{1n}\right)+k\epsilon^2\delta_{1n}$$

$$\alpha_{14}=\left[3k\left(\frac{1-\nu}{2}\right)\lambda_m^2\frac{\epsilon^2}{4}-k\epsilon^2\right](1-\delta_{4n})$$

$$\alpha_{15}=3k\left(\frac{1-\nu}{2}\right)\lambda_m^2\frac{\epsilon^2}{4}-k\epsilon^2$$

$$\alpha_{16}=-n-k\left(\frac{3-\nu}{2}\right)\lambda_m^2 n+\left[-\frac{\epsilon}{2}+k\left(\frac{3-\nu}{2}\right)\lambda_m^2\frac{\epsilon}{2}\right.$$

$$\left.+k\epsilon-k\left(\frac{\epsilon^3+4\epsilon}{4}\right)\right]\delta_{1n}-k\epsilon^2\left(\delta_{2n}+\frac{\epsilon}{4}\delta_{3n}\right)$$

$$\alpha_{17}=-\frac{n\epsilon}{2}-k\left(\frac{3-\nu}{2}\right)\lambda_m^2\frac{(n-2)\epsilon}{2}-k\frac{(\epsilon^3+4\epsilon)}{4}\ \ldots$$

$$-\left[\epsilon+k\frac{(\epsilon^3+4\epsilon)}{4}\right]\delta_{2n}-k\epsilon^2\left(\delta_{3n}+\frac{\epsilon}{4}\delta_{4n}\right)+k\epsilon(n-2)^2$$

$$\alpha_{18}=-\frac{n\epsilon}{2}-k\left(\frac{3-\nu}{2}\right)\lambda_m^2\frac{(n+2)\epsilon}{2}+k\frac{(\epsilon^3+4\epsilon)}{4}$$

$$-k\epsilon^2\left(\delta_{1n}+\frac{\epsilon}{4}\delta_{2n}\right)-k\epsilon(n+2)^2$$

$$\alpha_{19}=-k\epsilon^2\left(1+\delta_{4n}+\frac{\epsilon}{4}\delta_{5n}\right)\qquad \alpha_{20}=k\epsilon^2\left(1-\frac{\epsilon}{4}\delta_{1n}\right)$$

$$\alpha_{21}=-k\frac{\epsilon^3}{4}(1+\delta_{6n})\qquad \alpha_{22}=k\frac{\epsilon^3}{4}$$

$$\alpha_{23}=\nu\lambda_m\left(1+\frac{\epsilon}{2}\delta_{1n}\right)-k\left(\frac{1-\nu}{2}\right)\lambda_m n^2\left(1-\frac{\epsilon}{2}\delta_{1n}\right)$$

$$+k\lambda_m^3\left(1+\frac{\epsilon}{2}\delta_{1n}\right)$$

$$\alpha_{24}=\frac{\nu\epsilon\lambda_m}{2}(1+\delta_{2n})-k\left(\frac{1-\nu}{2}\right)\lambda_m\frac{\epsilon(n-2)n}{2}$$

$$+k\lambda_m^3\frac{\epsilon}{2}(1+\delta_{2n})$$

$$\alpha_{25}=\frac{\nu\epsilon\lambda_m}{2}-k\left(\frac{1-\nu}{2}\right)\lambda_m\frac{\epsilon(n+2)n}{2}+k\lambda_m^3\frac{\epsilon}{2}$$

$$\alpha_{26}=-n-k\left(\frac{3-\nu}{2}\right)\lambda_m^2 n+\left(-\frac{\epsilon}{2}+k\left(\frac{3-\nu}{2}\right)\lambda_m^2\frac{\epsilon}{2}\right.$$

$$\left.+k\epsilon-k\frac{(\epsilon^3+4\epsilon)}{4}\right)\delta_{1n}-k\epsilon^2\left(\delta_{2n}+\frac{\epsilon}{4}\delta_{3n}\right)$$

$$\alpha_{27}=-\frac{\epsilon(n-2)}{2}-k\left(\frac{3-\nu}{2}\right)\lambda_m^2\frac{n\epsilon}{2}(1-\delta_{2n})-k\epsilon n^2$$

$$+k\frac{(\epsilon^3+4\epsilon)}{4}(1-\delta_{2n})-k\epsilon^2\left(\delta_{3n}+\frac{\epsilon}{4}\delta_{4n}\right)+4k\epsilon\delta_{2n}$$

$$\alpha_{28}=-\frac{\epsilon(n+2)}{2}-k\left(\frac{3-\nu}{2}\right)\lambda_m^2\frac{n\epsilon}{2}+k\epsilon n^2$$

$$-k\frac{(\epsilon^3+4\epsilon)}{4}-k\epsilon^2\left(\delta_{1n}+\frac{\epsilon}{4}\delta_{2n}\right)$$

$$\alpha_{29}=k\epsilon^2\left(1-\delta_{4n}-\frac{\epsilon}{4}\delta_{5n}\right)\qquad \alpha_{30}=-k\epsilon^2\left(1+\frac{\epsilon}{4}\delta_{1n}\right)$$

$$\alpha_{31}=k\frac{\epsilon^3}{4}(1-\delta_{6n})\qquad \alpha_{32}=-k\frac{\epsilon^3}{4}$$

$$\alpha_{33}=k\lambda_m^4+2k\lambda_m^2 n^2+kn^4-2k\left(1+\frac{\epsilon^2}{2}\right)n^2$$

$$+\left(1+\frac{\epsilon^2}{2}\right)+k\left[\frac{(2+\epsilon^2)^2}{4}+2\epsilon^2+\frac{\epsilon^4}{8}\right]$$

$$+\left\{2k\epsilon+\epsilon+k\left[\epsilon(2+\epsilon^2)+\frac{\epsilon^3}{2}\right]\right\}\delta_{1n}$$

$$+\left(\frac{\epsilon^2}{4}+k\frac{\epsilon^2}{2}+k\frac{\epsilon^4}{4}\right)\delta_{2n}+k\frac{\epsilon^3}{2}\left(\delta_{3n}+\frac{\epsilon}{8}\delta_{4n}\right)$$

$$\alpha_{34}=-2k\epsilon n(n-2)+\epsilon\left(1+\delta_{2n}+\frac{\epsilon}{4}\delta_{3n}\right)-4k\epsilon(1+\delta_{2n})$$

$$+k\left[\epsilon(2+\epsilon^2)+\frac{\epsilon^3}{2}\right](1+\delta_{2n})+\left(-k\epsilon^2+k\frac{\epsilon^4}{4}\right)\delta_{3n}$$

$$+k\frac{\epsilon^3}{2}\left(\delta_{4n}+\frac{\epsilon}{8}\delta_{5n}\right)$$

$$\alpha_{35}=-2k\epsilon n(n+2)-4k\epsilon+\epsilon+k\left[\epsilon(2+\epsilon^2)+\frac{\epsilon^3}{2}\right]$$

$$+\left(\frac{\epsilon^2}{4}-k\epsilon^2+k\frac{\epsilon^4}{4}\right)\delta_{1n}+k\frac{\epsilon^3}{2}\left(\delta_{2n}+\frac{\epsilon}{8}\delta_{3n}\right)$$

$$\alpha_{36}=-k\frac{\epsilon^2}{2}(n-4)n+\frac{\epsilon^2}{4}(1+\delta_{4n})-4k\epsilon^2(1+\delta_{4n})$$

$$+k\left[\epsilon^2+(2+\epsilon^2)\frac{\epsilon^2}{4}\right](1+\delta_{4n})+\frac{k\epsilon^3}{2}\left(\delta_{5n}+\frac{\epsilon}{8}\delta_{6n}\right)$$

$$\alpha_{37}=-k\frac{\epsilon^2}{2}(n+4)n+\frac{\epsilon^2}{4}-4k\epsilon^2$$

$$+k\left[\epsilon^2+\frac{(\epsilon^2+2)\epsilon^2}{4}\right]+k\frac{\epsilon^3}{2}\left(\delta_{1n}+\frac{\epsilon}{8}\delta_{2n}\right)$$

$$\alpha_{38}=k\frac{\epsilon^3}{2}\left(1+\delta_{6n}+\frac{\epsilon}{8}\delta_{7n}\right)$$

$$\alpha_{39}=k\frac{\epsilon^3}{2}\left(1+\frac{\epsilon}{8}\delta_{1n}\right)$$

$$\alpha_{40}=k\frac{\epsilon^4}{16}(1+\delta_{8n})\qquad \alpha_{41}=k\frac{\epsilon^4}{16}$$

where

$$\delta_{in}=\begin{cases}0 & i\neq n\\ 1 & i=n\end{cases}$$

The coefficients b_i ($i=1,2,3...41$) are equal to

$$b_i=\bar{\alpha}_i\qquad (i=1,2,...,41)$$

except

$$b_6=-\bar{\alpha}_6,\quad b_{10}=-\bar{\alpha}_{10},\quad b_{16}=-\bar{\alpha}_{16},\quad b_{17}=-\bar{\alpha}_{17},$$
$$b_{18}=-\bar{\alpha}_{18},\quad b_{19}=-\bar{\alpha}_{19},\quad b_{20}=-\bar{\alpha}_{20},\quad b_{21}=-\bar{\alpha}_{21},$$
$$b_{22}=-\bar{\alpha}_{22},\quad b_{26}=-\bar{\alpha}_{26},\quad b_{27}=-\bar{\alpha}_{27},\quad b_{28}=-\bar{\alpha}_{28},$$
$$b_{29}=-\bar{\alpha}_{29},\quad b_{30}=-\bar{\alpha}_{30},\quad b_{31}=-\bar{\alpha}_{31},\quad b_{32}=-\bar{\alpha}_{32}$$

where $\bar{\alpha}_i$ ($i=1,2,3...41$) is obtained from α_i ($i=1,2,3...41$) by replacing (δ_{in}) with $(-\delta_{in})$ in the latter.

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